$$U = \frac{1}{2}(\sigma_x e_x + \sigma_y e_y + \sigma_z e_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) = \frac{1}{2}\sigma_{ij}e_{ij} \tag{6.1.7}$$

Although the preceding results were developed for the case of uniform stress with no body forces, it can be shown (see Exercise 6-1) that identical results are found if body forces are included and the stresses are allowed to vary continuously. The total strain energy stored in an elastic solid occupying a region V is then given by the integral over the domain

$$U_T = \iiint_V U dxdydz \tag{6.1.8}$$

Using Hooke's law, the stresses can be eliminated from relation (6.1.7) and the strain energy can be expressed solely in terms of strain. For the isotropic case, this result becomes

$$\begin{aligned} U(\boldsymbol{e}) &= \frac{1}{2}\lambda e_{jj}e_{kk} + \mu e_{ij}e_{ij} \\ &= \frac{1}{2}\lambda(e_x + e_y + e_z)^2 + \mu\left(e_x^2 + e_y^2 + e_z^2 + \frac{1}{2}\gamma_{xy}^2 + \frac{1}{2}\gamma_{yz}^2 + \frac{1}{2}\gamma_{zx}^2\right) \end{aligned} \tag{6.1.9}$$

Likewise, the strains can be eliminated and the strain energy can be written in terms of stress

$$\begin{aligned} U(\boldsymbol{\sigma}) &= \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{jj}\sigma_{kk} \\ &= \frac{1+\nu}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2) - \frac{\nu}{2E}(\sigma_x + \sigma_y + \sigma_z)^2 \end{aligned} \tag{6.1.10}$$

After reviewing the various developed forms in terms of the stresses or strains, it is observed that the strain energy is a *positive definite quadratic form* with the property

$$U \geq 0 \tag{6.1.11}$$

for all values of σ_{ij} and e_{ij}, with equality only for the case with $\sigma_{ij} = 0$ or $e_{ij} = 0$. Actually, relation (6.1.11) is valid for all elastic materials, including both isotropic and anisotropic solids.

For the uniaxial deformation case, by using relation (6.1.4) note that the derivative of the strain energy in terms of strain yields

$$\frac{\partial U(\boldsymbol{e})}{\partial e_x} = \frac{\partial}{\partial e_x}\left(\frac{Ee_x^2}{2}\right) = Ee_x = \sigma_x$$

and likewise

$$\frac{\partial U(\boldsymbol{\sigma})}{\partial \sigma_x} = \frac{\partial}{\partial \sigma_x}\left(\frac{\sigma_x^2}{2E}\right) = \frac{\sigma_x}{E} = e_x$$

These specific uniaxial results can be generalized (see Exercise 6-4) for the three-dimensional case, giving the relations problem solution is unique. Note that if tractions are prescribed over the entire boundary, then $u_i^{(1)}$ and $u_i^{(2)}$ may differ by rigid-body motion.

6.3 Bounds on the Elastic Constants

Strain energy concepts allow us to generate particular bounds on elastic constants. For the isotropic case, consider the following three stress states previously investigated in Section 4.3.

6.3.1 Uniaxial Tension

Uniform uniaxial deformation in the x direction is given by the stress state

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.3.1}$$

For this case, the strain energy reduces to

$$U = \frac{1+v}{2E}\sigma^2 - \frac{v}{2E}\sigma^2 = \frac{\sigma^2}{2E} \tag{6.3.2}$$

Because the strain energy is positive definite, relation (6.3.2) implies that the modulus of elasticity must be positive

$$E > 0 \tag{6.3.3}$$

6.3.2 Simple Shear

Consider next the case of uniform simple shear defined by the stress tensor

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.3.4}$$

The strain energy becomes

$$U = \frac{1+v}{2E}(2\tau^2) = \frac{\tau^2}{E}(1+v) \tag{6.3.5}$$

Again, invoking the positive definite property of the strain energy and using the previous result of $E > 0$ gives

$$1 + v > 0 \Rightarrow v > -1 \tag{6.3.6}$$

6.3.3 Hydrostatic Compression

The final example is chosen as uniform hydrostatic compression specified by

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \tag{6.3.7}$$

Note that hydrostatic tension could also be used for this example. Evaluating the strain energy yields

$$U = \frac{1+v}{2E} 3p^2 - \frac{v}{2E}(-3p)^2 = \frac{3p^2}{2E}(1-2v) \tag{6.3.8}$$

Using the positive definite property with $E > 0$ gives the result

$$1 - 2v > 0 \Rightarrow v < \frac{1}{2} \tag{6.3.9}$$

Combining relations (6.3.6) and (6.3.9) places the following bounds on Poisson's ratio:

$$-1 < v < \frac{1}{2} \tag{6.3.10}$$

Using relations between the elastic constants given in Table 4-1, the previous results also imply that

$$k > 0, \ \mu > 0 \tag{6.3.11}$$

Experimental evidence indicates that most real materials have positive values of Poisson's ratio, and thus $0 < v < 1/2$. This further implies that $\lambda > 0$. Bounds on elastic moduli for the anisotropic case are more involved and are discussed in Chapter 11.

6.4 Related Integral Theorems

Within the context of linear elasticity, several integral relations based on work and energy can be developed. We now wish to investigate three particular results referred to as *Clapeyron's theorem, Betti's reciprocal theorem*, and *Somigliana's identity*.

6.4.1 Clapeyron's Theorem

The strain energy of an elastic solid in static equilibrium is equal to one-half the work done by the external body forces F_i and surface tractions T_i^n

$$2\int_V U dV = \int_S T_i^n u_i dS + \int_V F_i u_i dV \tag{6.4.1}$$

The proof of this theorem follows directly from results in relation (6.2.4).

6.4.2 Betti/Rayleigh Reciprocal Theorem

If an elastic body is subject to two body and surface force systems, then the work done by the first system of forces $\{\boldsymbol{T}^{(1)}, \boldsymbol{F}^{(1)}\}$ acting through the displacements $\boldsymbol{u}^{(2)}$ of the second system is equal to the work done by the second system of forces $\{\boldsymbol{T}^{(2)}, \boldsymbol{F}^{(2)}\}$ acting through the displacements $\boldsymbol{u}^{(1)}$ of the first system; that is: